Modern Mathematics: Lecture Summary

Advanced Concepts in Abstract Algebra and Topology

1 Introduction to Abstract Structures

Modern mathematics shifts the focus from specific numbers and geometric figures to generalized abstract structures. At the heart of this study lie algebraic structures equipped with operations satisfying specific axioms.

Definition 1.1 (Group). *A group $(G, \cdot)$ is a set G combined with a binary operation $\cdot$ that satisfies three axioms:*

- ***Associativity:*** *For all $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.*
- ***Identity Element:*** *There exists an element $e \in G$ such that for every $a \in G$, $e \cdot a = a \cdot e = a$.*
- ***Inverse Element:*** *For each $a \in G$, there exists an element $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.*

2 Category Theory Foundations

Often referred to as "the mathematics of mathematics," category theory provides a universal language to describe mathematical structures and relationships between them.

Theorem 2.1 (Functorial Composition). *Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories. If $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ are functors, then the composition $G \circ F : \mathcal{C} \to \mathcal{E}$ is also a functor, defined by:*

$$(G \circ F)(A) = G(F(A)) \quad \textit{for objects } A \in \textit{obj}(\mathcal{C})$$

and similarly for morphisms.

3 Topological Spaces

Topology generalizes the notions of continuity, connectedness, and convergence from metric spaces to more abstract settings.

Definition 3.1 (Topology). *Let X be a set. A topology on X is a collection τ of subsets of X satisfying:*

1. *The empty set $\emptyset$ and X are in τ.*
2. *The union of any collection of sets in τ is in τ.*
3. *The intersection of any finite collection of sets in τ is in τ.*

The pair (X, τ) is called a topological space.