Ultimate Calculus Cheat Sheet

Derivatives, Integrals, Series, and Vector Calculus

1. Limits and Continuity

Definition: $\lim_{x\to a} f(x) = L$ means $f(x)$ approaches L as x approaches a.

L'Hôspital's Rule: If $\lim \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Squeeze Theorem: If $g(x) \le f(x) \le h(x)$ and $\lim g(x) = \lim h(x) = L$, then $\lim f(x) = L$.

2. Derivatives

Definition:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Basic Rules:

- **Power:** $\frac{d}{dx}[x^n] = nx^{n-1}$
- **Product:** $(fg)' = f'g + fg'$
- **Quotient:** $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$
- **Chain:** $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$

Common Derivatives:

- $\frac{d}{dx}[e^x] = e^x$, $\frac{d}{dx}[a^x] = a^x \ln a$
- $\frac{d}{dx}[\ln x] = \frac{1}{x}$
- $\frac{d}{dx}[\sin x] = \cos x$, $\frac{d}{dx}[\cos x] = -\sin x$
- $\frac{d}{dx}[\tan x] = \sec^2 x$
- $\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$

3. Integrals

Fundamental Theorem of Calculus:

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{where } F'(x) = f(x)$$

Integration by Parts:

$$\int u\,dv = uv - \int v\,du$$

Common Integrals:

- $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
- $\int \frac{1}{x}\,dx = \ln|x| + C$
- $\int e^x\,dx = e^x + C$
- $\int \sin x\,dx = -\cos x + C$
- $\int \cos x\,dx = \sin x + C$
- $\int \sec^2 x\,dx = \tan x + C$
- $\int \frac{1}{1+x^2}\,dx = \arctan x + C$
- $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$

4. Applications of Integration

Arc Length:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

Surface Area (about x-axis):

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx$$

Volume (Disk Method):

$$V = \pi \int_a^b [f(x)]^2\,dx$$

5. Sequences & Series

Geometric Series: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $|r| < 1$.

p-Series: $\sum \frac{1}{n^p}$ converges if $p > 1$.

Taylor Series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Common Maclaurin Series:

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \dots$
- $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$
- $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$
- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (|x| < 1)$

6. Vector Calculus

Gradient: $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

Divergence: $\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

Curl: $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$

Green's Theorem:

$$\oint_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$