Modern Mathematics: Abstract Algebra & Topology Overview

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Abstract

This lesson summary explores foundational pillars of modern mathematics, focusing on abstract algebraic structures (groups, rings, fields) and basic topological concepts. Designed for advanced candidates, this document synthesizes core definitions and theorems with rigorous precision.

1 Abstract Algebra

Modern algebra shifts the focus from specific numbers and arithmetic operations to generalized abstract structures.

Definition 1.1 (Group). *A group $(G, \cdot)$ is a set G equipped with a binary operation $\cdot$ satisfying:*

1. ***Associativity:*** $\forall a, b, c \in G, \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$.
2. ***Identity Element:*** $\exists e \in G$ *such that* $\forall a \in G, \quad e \cdot a = a \cdot e = a$.
3. ***Inverse Element:*** $\forall a \in G, \exists a^{-1} \in G$ *such that* $a \cdot a^{-1} = a^{-1} \cdot a = e$.

Theorem 1.2 (Lagrange's Theorem). *If G is a finite group and H is a subgroup of G, then the order of H divides the order of G:*

$$|H| \mid |G|$$

2 Topology

Topology generalizes the notions of continuity, compactness, and connectedness from metric spaces to arbitrary sets.

Definition 2.1 (Topological Space). *A topological space is an ordered pair (X, τ), where X is a set and τ is a collection of subsets of X (called open sets) satisfying:*

1. $\emptyset, X \in \tau$.
2. *The union of any collection of sets in τ is in τ.*
3. *The intersection of any finite collection of sets in τ is in τ.*

3 Visualizing Algebraic Lattices

Below is a structural schematic representing a subgroup lattice diagram generated via TikZ:

$$
\begin{tikzcd}[column sep=small]
 & G \text{ (Symmetric Group } S_3\text{)} \arrow[dl, no head] \arrow[dr, no head] & \\
A_3 \text{ (Alternating)} \arrow[dr, no head] & & C_2 \text{ (Subgroup)} \arrow[dl, no head] \\
 & \{e\} &
\end{tikzcd}
$$