Ultimate Cheat Sheet: Inequalities

Master the core concepts, properties, and advanced problem-solving techniques.

1. Fundamental Properties

$a, b, c, d \in \mathbb{R}$.

- **Trichotomy:** Exactly one holds: $a < b$, $a = b$, or $a > b$.
- **Transitivity:** If $a < b$ and $b < c$, then $a < c$.
- **Addition:** If $a < b$, then $a + c < b + c$.
- **Multiplication (Positive):** If $a < b$ and $c > 0$, then $ac < bc$.
- **Multiplication (Negative):** If $a < b$ and $c < 0$, then $ac > bc$ (reverses inequality sign).
- **Reciprocal:** If $0 < a < b$, then $\frac{1}{a} > \frac{1}{b}$.

2. Absolute Value Inequalities

$c > 0$:

- $|x| < c \iff -c < x < c$
- $|x| \leq c \iff -c \leq x \leq c$
- $|x| > c \iff x < -c$ or $x > c$
- $|x| \geq c \iff x \leq -c$ or $x \geq c$
- **Triangle Inequality:**

$$|x + y| \leq |x| + |y|$$

$$||x| - |y|| \leq |x - y|$$

3. Polynomial Rational Inequalities

Method of Intervals (Wavy Line Method):

1. Move all terms to one side so the other side is 0 (e.g., $P(x) > 0$).
2. Factor the polynomial completely into linear and irreducible quadratic factors.
3. Find the critical points (roots where $P(x) = 0$ or undefined).
4. Plot critical points on a number line.
5. Test intervals starting from the right with a positive sign, alternating if multiplicity is odd.

4. Classical Means Inequalities

non-negative real numbers $x_1, x_2, \ldots, x_n$:

- **AM-GM Inequality**:

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}$$

  Equality holds iff $x_1 = x_2 = \cdots = x_n$.

- **GM-HM Inequality**:

$$\sqrt[n]{x_1 \ldots x_n} \geq \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}$$

- **RMS-AM-GM-HM Chain**:

$$\text{RMS} \geq \text{AM} \geq \text{GM} \geq \text{HM}$$

  where $\text{RMS} = \sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}}$.

5. Cauchy-Schwarz Inequality

any real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$:

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

Engel Form (Titu's Lemma):

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n}$$

$b_i > 0$.

6. Bernoulli's Inequality

any real number $x \geq -1$ and integer $n \geq 0$:

$$(1 + x)^n \geq 1 + nx$$

$n \geq 1$ and $x \geq -1, x \neq 0$:

$$(1 + x)^n > 1 + nx \quad (n \geq 2)$$

7. Rearrangement Inequality

$a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$.any permutation $(x_1, \ldots, x_n)$ of $(b_1, \ldots, b_n)$:

$$a_n b_1 + \cdots + a_1 b_n \leq \sum_{i=1}^{n} a_i x_i \leq a_1 b_1 + \cdots + a_n b_n$$

- Maximum is achieved with sorted order (like-sorted).
- Minimum is achieved with reverse-sorted order.