Masterclass Summary: Advanced Foundations of Arithmetic

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Abstract

This lecture summary covers the rigorous foundations of arithmetic, exploring the algebraic properties of numbers, divisibility theory, fundamental theorem of arithmetic, and modular congruences. Designed for elite scholars seeking a deep, axiomatic understanding of elementary operations.

1 Axiomatic Foundations of Numbers

Arithmetic begins with the natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$, extended through the integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$, and real numbers $\mathbb{R}$. The fundamental operations of addition $(+)$ and multiplication $(\times)$ satisfy the field axioms:

$$\begin{aligned} \text{Associativity:} \quad & (a+b)+c = a+(b+c) \\ \text{Commutativity:} \quad & a+b = b+a \\ \text{Distributivity:} \quad & a \cdot (b+c) = a \cdot b + a \cdot c \end{aligned}$$

2 Divisibility and the Euclidean Algorithm

An integer a is divisible by a non-zero integer b (denoted $b \mid a$) if there exists an integer k such that $a = bk$.

The **Division Algorithm** states that for any integers a and b with $b > 0$, there exist unique integers q and r such that:

$$a = bq + r, \quad 0 \le r < b$$

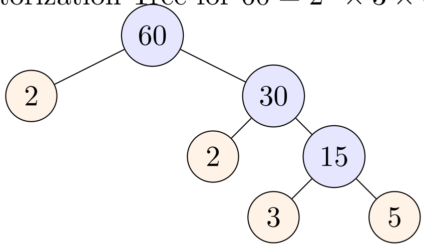
The Greatest Common Divisor ($\gcd(a, b)$) can be efficiently computed using the Euclidean Algorithm, relying on the principle that $\gcd(a, b) = \gcd(b, a \pmod b)$.

3 The Fundamental Theorem of Arithmetic

Every integer $n > 1$ can be represented uniquely as a product of prime numbers, up to the order of the factors. Let $p_1 < p_2 < \cdots < p_k$ be distinct primes and $a_i \ge 1$ be integers:

$$n = \prod_{i=1}^{k} p_i^{a_i} = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$$

Prime Factorization Tree for $60 = 2^2 \times 3 \times 5$



4 Modular Arithmetic

Let m be a positive integer called the modulus. We say that a is congruent to b modulo m, written as:

$$a \equiv b \pmod{m} \iff m \mid (a - b)$$

This equivalence relation preserves sums and products, serving as the backbone for modern cryptography and number theory.